THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2230A Complex Variables with Applications 2017-2018
Suggested Solution to final examination

1. (a) Given $f(z) = \dfrac{2}{(1-z)^3}$ and $0 < r < 1$.

For $|z| = r$, since $|1-z|^3 \geq (1-|z|)^3 = (1-r)^3$, we have

$$|f(z)| \leq \frac{2}{(1-r)^3} \quad \forall |z| = r$$

Furthermore, $|f(r)| = \dfrac{2}{(1-r)^3}$. Therefore,

$$\max_{|z|=r} |f(z)| = \frac{2}{(1-r)^3}$$

(b) By computing $f'(z)$, $f''(z)$ and $f'''(z)$, one can find that $f^{(n)}(z)$ is given by

$$f^{(n)}(z) = \frac{(n+2)!}{(1-z)^{n+3}} \quad \forall n \in \mathbb{N}$$

(c) By Cauchy's inequality,

$$(n+2)! = |f^{(n)}(0)| \leq \frac{n! \max_{|z|=r} |f(z)|}{r^n} = \frac{n! \cdot 2}{r^n (1-r)^3}$$

Therefore,

$$r^n (1-r)^3 \leq \frac{2}{(n+2)(n+1)}$$

2. (a) First of all, in the given branch, $\log(-1) = \log e^{i\pi} = i\pi$.

Furthermore, by computing several derivatives, one can observe that

$$\log^{(n)} z = \frac{(-1)^{n+1}(n-1)!}{z^n} \quad \forall n \in \mathbb{N}$$

So

$$\log^{(n)}(-1) = \frac{(-1)^{n+1}(n-1)!}{(-1)^n} = -(n-1)! \quad \forall n \in \mathbb{N}$$

As a result, for $0 \leq |z+1| < 1$, we have

$$\begin{aligned} \log z &= i\pi + \sum_{n=1}^{\infty} \frac{-(n-1)!}{n!}(z+1)^n \\ &= i\pi - \sum_{n=1}^{\infty} \frac{1}{n}(z+1)^n \end{aligned}$$

(b) Note that $\sin \pi z = -\sin(\pi(z+1)) = -\left(\pi(z+1) - \frac{\pi^3(z+1)^3}{3!} + \dots\right)$.

Hence

$$\begin{aligned} f(z) &= \frac{i\pi - (z+1) - \frac{1}{2}(z+1)^2 + \dots}{-\left(\pi(z+1) - \frac{\pi^3(z+1)^3}{3!} + \dots\right)} \\ &= \frac{i\pi - (z+1) - \frac{1}{2}(z+1)^2 + \dots}{-\pi(z+1)} \times \frac{1}{1 - \frac{\pi^2(z+1)^2}{3!} + \dots} \\ &= \left(\frac{-i}{z+1} + \frac{1}{\pi} + \frac{1}{2\pi}(z+1) + \dots\right)\left(1 + \frac{\pi^2(z+1)^2}{3!} + \dots\right) \\ &= -\frac{i}{z+1} + \frac{1}{\pi} + \left(\frac{1}{2\pi} - \frac{i\pi^2}{3!}\right)(z+1) + \dots \end{aligned}$$

This gives the coefficients of $(z+1)^{-1}, z^0$ and $(z+1)^1$.

(Alternatively, you may use long division to do this question.)

(c) By (b), we have $\text{Res}_{z_0=-1} f(z) = -i$.

(Alternatively, you may use any other ways to find out the residue.)

3. Consider $f(z) = \dfrac{z^2}{(z^2+4)(z^2+16)}$ and the simple closed contour $\Gamma_R = [-R, R] \cup C_R^+$ with $R > 4$.

$f(z)$ has singular points at $z = \pm 2i, \pm 4i$. By Cauchy's residue theorem, we have

$$\int_{\Gamma_R} f(z)dz = 2\pi i \left(\text{Res}_{z=2i} f(z) + \text{Res}_{z=4i} f(z)\right)$$

At $z = 2i$, $\text{Res}_{z=2i} f(z) = \text{Res}_{z=2i} \dfrac{(z^2)/(z+2i)(z^2+16)}{(z-2i)} = \dfrac{((2i)^2)}{((2i)+2i)((2i)^2+16)} = \dfrac{i}{12}$

At $z = 4i$, $\text{Res}_{z=4i} f(z) = \text{Res}_{z=4i} \dfrac{(z^2)/(z^2+4)(z+4i)}{(z-4i)} = \dfrac{((4i)^2)}{((4i)^2+4)(4i+4i)} = -\dfrac{i}{6}$

Hence

$$\int_{\Gamma_R} f(z)dz = 2\pi i\left(\frac{i}{12} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

Note that

$$\int_{\Gamma_R} f(z)dz = \int_{-R}^{R} \frac{x^2}{(x^2+4)(x^2+16)}dx + \int_{C_R^+} \frac{z^2}{(z^2+4)(z^2+16)}dz$$

Also,

$$\left|\int_{C_R^+} \frac{z^2}{(z^2+4)(z^2+16)}dz\right| \le \frac{R^2}{(R^2-4)(R^2-16)}\pi R = \frac{\pi R^3}{(R^2-4)(R^2-16)} \to 0$$

as $R \to 0$.

As a result,

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+16)}dx = \frac{\pi}{6}$$

Since the integrand is an even function, we have

$$\int_{0}^{\infty} \frac{x^2}{(x^2+4)(x^2+16)}dx = \frac{\pi}{12}$$

4. Given that $f(z) = \dfrac{z^{n-1}e^{P(\frac{1}{z})}}{P(z)}$. Note that $f(z)$ is analytic except at $z = 0$ and roots of $P(z)$. Therefore, for large $R > 0$,

$$\begin{aligned}
\int_{C_R} \frac{z^{n-1}e^{P(\frac{1}{z})}}{P(z)} dz &= 2\pi i \,\mathrm{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \\
&= 2\pi i \,\mathrm{Res}_{z=0} \left[\frac{1}{z^2} \frac{\left(\frac{1}{z}\right)^{n-1} e^{P(z)}}{P\left(\frac{1}{z}\right)}\right] \\
&= 2\pi i \,\mathrm{Res}_{z=0} \left[\frac{1}{z^{n+1}} \times \frac{e^{P(z)}}{\frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \cdots + a_0}\right] \\
&= 2\pi i \,\mathrm{Res}_{z=0} \left[\frac{1}{z} \times \frac{e^{P(z)}}{a_n + a_{n-1}z + \cdots + a_0 z^n}\right]
\end{aligned}$$

Since the function $\left(\dfrac{e^{P(z)}}{a_n + a_{n-1}z + \cdots + a_0 z^n}\right)$ is analytic at $z = 0$ and $\dfrac{e^{P(0)}}{a_n} = \dfrac{e^{a_0}}{a_n} \neq 0$, we have

$$\int_{C_R} \frac{z^{n-1}e^{P(\frac{1}{z})}}{P(z)} dz = 2\pi i \,\mathrm{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i \left(\frac{e^{a_0}}{a_n}\right)$$